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FINITE HORIZON H_∞ WITH PARAMETER VARIATIONS

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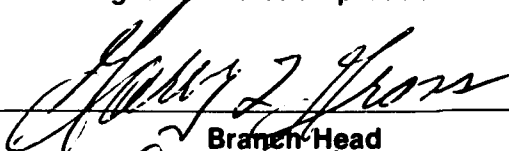
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13. ABSTRACT (Maximum 200 words) In this technical note we consider the finite horizon H_{∞} performance robustness problem with parameter variations. Assuming a nominal model, an iterative procedure is given to synthesize a suboptimal H_{∞} controller, which yields the required performance even under parameter variations. As a by-product, an expression for the variation of performance owing to parameter variations is given. An example which illustrates the methodology is worked out under parameter uncertainties.					
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FINITE HORIZON H_{∞} WITH PARAMETER VARIATIONS

1. INTRODUCTION

The suboptimal H_{∞} problem can be solved via the the solution of two algebraic Riccati equations (Doyle et al., 1989). In the finite horizon case, expressions for a suboptimal controller have also been derived in the time-varying setting (Subrahmanyam 1992a), without resorting to various transformations, such as the ones given in (Safonov et al., 1989). In the time-invariant case, the solutions of the dynamic Riccati equations eventually converge to constant matrices. Also, efficient algorithms for the computation of the infimal H_{∞} norm have been given in the finite horizon case (Subrahmanyam 1992b,c,d). Our computational experience indicates that for time-invariant systems, the infimal H_{∞} norm can be very nearly approached using full state feedback, whereas with output feedback, the suboptimal value that gives a viable controller is generally much higher than the infimal value (Subrahmanyam 1992e).

It is difficult to design a suboptimal controller taking into account parameter variations in the system matrices. It is doubtful whether a noniterative solution exists to the performance robustness problem. The reason for this is the fact that when one designs a suboptimal H_{∞} controller at the nominal values of the parameters, the performance of the controller under a variation in the system matrices is a priori unknown. In fact, in our treatment here, knowledge of the controller is essential for the computation of the variation of performance under variations in the system matrices. It is tacitly assumed that the controller and observer matrices designed at a nominal point are fixed even under parameter uncertainties since we have no way to measure the variations in the system matrices in general.

To solve the performance robustness problem, we need to have an idea of the variations in performance. To this end, we derive an expression for the variation in performance for a specific controller in terms of variations in the system matrices. If the performance degradation for the range of parameter variations is unsatisfactory, we need to redesign the suboptimal controller.

The technical note is organized as follows. In Section 2 we formulate the finite horizon H_∞ performance robustness problem. Section 3 presents a summary of the equations involved in the design of a suboptimal H_∞ controller. Section 4 presents a procedure for the computation of actual performance of the suboptimal controller. In Section 5, a formula for the degradation in performance owing to parameter variations is given. Section 6 presents an iterative procedure for the design of a suboptimal controller which has adequate performance robustness. An example of the control system design which illustrates the usefulness of the theory is given in Section 7.

2. PROBLEM FORMULATION

Let the n -dimensional time-varying system be given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(t_0) = 0, \quad (2.1)$$

$$z = C(t)x + D(t)u + E(t)v, \quad (2.2)$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v. \quad (2.3)$$

Without loss of generality, let $t_0 = 0$. Also let

$$\lambda_{opt} = \max_u \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt}, \quad (2.4)$$

where R and W are assumed to be positive definite and the superscript $*$ denotes a matrix or vector transpose. Computational techniques for the evaluation of λ_{opt} are given in the papers by Subrahmanyam (1992b,c,d). The suboptimal H_∞ problem can be stated as follows. Given $\lambda < \lambda_{opt}$, find full state and output feedback controllers, if these exist, for which

$$\min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt} > \lambda. \quad (2.5)$$

The above problem has been solved in Subrahmanyam (1992a) and a summary of the design equations will be given in Section 3.

The suboptimal performance robustness problem can be stated as follows. Given $\bar{\lambda} < \lambda_{opt}$, find state and output feedback controllers, if these exist, for which

$$\min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt} > \bar{\lambda},$$

even under specified variations in the system matrices in (2.1)-(2.3).

3. FEEDBACK SOLUTIONS

In this section, we give equations for a suboptimal H_∞ design. For the details of derivation, see Subrahmanyam (1992a,1991).

The system is given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(0) = 0, \quad (3.1)$$

$$z = C(t)x + D(t)u + E(t)v, \quad (3.2)$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v. \quad (3.3)$$

Let

$$\lambda_{opt} = \max_u \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt}, \quad (3.4)$$

and let $\lambda < \lambda_{opt}$. Also, let W_1, \dots, W_6 be defined by

$$z^* W z = x^* W_1 x + 2x^* W_2 u + u^* W_3 u + 2x^* W_4 v + v^* W_5 v + 2u^* W_6 v. \quad (3.5)$$

Let

$$\tilde{z}_1 = B_2^* \tilde{e} + E_2^* \tilde{u} + E^* \tilde{w}, \quad (3.6)$$

and let $\tilde{W}_1, \dots, \tilde{W}_6$ be defined by

$$\tilde{z}_1^* R^{-1} \tilde{z}_1 = \tilde{e}^* \tilde{W}_1 \tilde{e} + 2\tilde{e}^* \tilde{W}_2 \tilde{u} + \tilde{u}^* \tilde{W}_3 \tilde{u} + 2\tilde{e}^* \tilde{W}_4 \tilde{w} + \tilde{w}^* \tilde{W}_5 \tilde{w} + 2\tilde{u}^* \tilde{W}_6 \tilde{w}. \quad (3.7)$$

We make the following assumptions.

- (a) Let the final time T satisfy the assumption in Lemma 4.2 of Subrahmanyam 1992a.
- (b) Assume that for each $t \in [0, T]$, $R - \lambda W_5$ is positive definite and $W_3 + \lambda W_6(R - \lambda W_5)^{-1} W_6^*$ is invertible.
- (c) Also, assume that for each $t \in [0, T]$, $W^{-1} - \lambda W_5$ is positive definite and $\tilde{W}_3 + \lambda \tilde{W}_6(W^{-1} - \lambda \tilde{W}_5)^{-1} \tilde{W}_6^*$ is invertible.

The relevant controller equations are

$$\Omega = (R - \lambda W_5)^{-1}, \quad (3.8)$$

$$\Lambda = \{W_3 + \lambda W_6 \Omega W_6^*\}^{-1}, \quad (3.9)$$

$$U_1 = \Lambda(B_1^* + \lambda W_6 \Omega B_2^*), \quad (3.10)$$

$$U_2 = -\Lambda(W_2^* + \lambda W_6 \Omega W_4^*), \quad (3.11)$$

$$V_1 = \lambda \Omega(-B_2^* + W_6^* U_1), \quad (3.12)$$

$$V_2 = \lambda \Omega(W_4^* + W_6^* U_2), \quad (3.13)$$

$$\begin{aligned} \dot{P} + P(A + B_1 U_2 + B_2 V_2) + (A^* - W_2 U_1 - W_4 V_1)P \\ + P(B_1 U_1 + B_2 V_1)P - (W_1 + W_2 U_2 + W_4 V_2) = 0, \quad P(T) = 0. \end{aligned} \quad (3.14)$$

Note that the above equation is symmetric.

The relevant observer equations are

$$\Phi = (W^{-1} - \lambda \tilde{W}_5)^{-1}, \quad (3.15)$$

$$\Gamma = \{\tilde{W}_3 + \lambda \tilde{W}_6 \Phi \tilde{W}_6^*\}^{-1}, \quad (3.16)$$

$$\tilde{A} = A + B_2 V_1 P + B_2 V_2, \quad (3.17)$$

$$\tilde{B} = -(U_1 P + U_2), \quad (3.18)$$

$$\tilde{C} = C_2 + E_2 V_1 P + E_2 V_2, \quad (3.19)$$

$$S_1 = \Gamma(\tilde{C} + \lambda \tilde{W}_6 \Phi D \tilde{B}), \quad (3.20)$$

$$S_2 = -\Gamma(\tilde{W}_2^* + \lambda \tilde{W}_6 \Phi \tilde{W}_4^*), \quad (3.21)$$

$$T_1 = \lambda \Phi(-D \tilde{B} + \tilde{W}_6^* S_1), \quad (3.22)$$

$$T_2 = \lambda \Phi(\tilde{W}_4^* + \tilde{W}_6^* S_2). \quad (3.23)$$

$$\begin{aligned} \dot{Y} = (\tilde{A} - \tilde{W}_2 S_1 - \tilde{W}_4 T_1)Y + Y(\tilde{A}^* + \tilde{C}^* S_2 + \tilde{B}^* D^* T_2) \\ + Y(\tilde{C}^* S_1 + \tilde{B}^* D^* T_1)Y - (\tilde{W}_1 + \tilde{W}_2 S_2 + \tilde{W}_4 T_2), \quad Y(0) = 0. \end{aligned} \quad (3.24)$$

Note that the above equation is symmetric.

A suboptimal controller is given by

$$\dot{q} = Aq + B_1(U_1 P q + U_2 q) + B_2(V_1 P q + V_2 q) + L(\tilde{C}q + D_2 u - y), \quad (3.25)$$

$$L = (S_1 Y + S_2)^*, \quad (3.26)$$

$$u = (U_1 P + U_2)q. \quad (3.27)$$

In the case of time-invariant systems, the solutions of the Riccati equations (3.14) and (3.24) eventually converge to constant matrices. Note that the full state feedback controller is simply given by $u = (U_1P + U_2)x$.

4. COMPUTATION OF PERFORMANCE

We consider the output feedback case. The full state feedback case is covered by the ensuing analysis as well. The closed loop system is given by

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} A & B_1(U_1P + U_2) \\ -LC_2 & \tilde{U} - LD_2(U_1P + U_2) \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} B_2 \\ -LE_2 \end{pmatrix} v, \quad (4.1)$$

where

$$\tilde{U} = A + B_1(U_1P + U_2) + B_2(V_1P + V_2) + L(\tilde{C} + D_2(U_1P + U_2)). \quad (4.2)$$

Note that all the matrices can be time-varying. We now specialize to the case where

$$x(0) = q(0) = 0. \quad (4.3)$$

Let

$$z = Cx + D(U_1P + U_2)q + Ev. \quad (4.4)$$

For the controller

$$u = (U_1P + U_2)q, \quad (4.5)$$

the performance is given by

$$\min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt}. \quad (4.6)$$

The above value is strictly greater than λ by our design procedure. For time-invariant systems, it is convenient to consider constant solutions of the two dynamic Riccati equations

and take the matrices $U_1P + U_2$ and $S_1Y + S_2$ to be fixed. In some cases, it is possible that the performance may be below λ because of the disregard for the time dependence. Nevertheless, in such cases, the performance can be artificially improved by designing the controller at a higher value of λ than the one specified.

We now describe a procedure to compute the value of (4.6). Let (4.1) be rewritten as

$$\dot{x}_s = A_s x_s + B_s v, \quad x_s(0) = 0, \quad (4.7)$$

where v needs to be chosen to minimize

$$\frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \left\{ \frac{1}{2} x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2} v^* Q_3 v \right\} dt}. \quad (4.8)$$

Note that the denominator of (4.6) can put in the form of the denominator of (4.8) by virtue of (4.4).

This problem has previously been solved by Subrahmanyam (1992f, 1990) and conditions guaranteeing the existence of a minimizing v are also reported in these references. For the sake of completeness, we present here the details concerning the characterization of a minimizing v and the computation of the performance corresponding to the minimizing v .

In the following theorem, we give conditions that are satisfied by an optimal v which minimizes (4.8) subject to (4.7).

THEOREM 4.1. *Consider the system given by (4.7) and (4.8). Let*

$$\hat{\lambda} = \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v \, dt}{\int_0^T \left\{ \frac{1}{2} x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2} v^* Q_3 v \right\} dt}. \quad (4.9)$$

Assume that $(R(t) - \hat{\lambda}Q_3(t))^{-1}$ exists for all $t \in [0, T]$. If (x_s, v) is optimal, then there exists a nonzero $\rho(t)$ such that

$$\frac{d\rho}{dt} = -A_s^* \rho - \hat{\lambda}Q_1 x_s - \hat{\lambda}Q_2 v, \quad \rho(T) = 0, \quad (4.10)$$

and

$$v(t) = (R - \hat{\lambda}Q_3)^{-1} \{ \hat{\lambda}Q_2^* x_s + B_s^* \rho \}. \quad (4.11)$$

Proof. If v minimizes (4.8), then it also minimizes

$$\tilde{J}(v) \triangleq \int_0^T \frac{1}{2} v^* R v dt - \hat{\lambda} \int_0^T \left\{ \frac{1}{2} x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2} v^* Q_3 v \right\} dt. \quad (4.12)$$

The theorem now follows from the maximum principle (Pontryagin et al., 1962). \square

Let

$$\tilde{A} = A_s + \hat{\lambda}B_s(R - \hat{\lambda}Q_3)^{-1}Q_2^*, \quad (4.13)$$

$$\tilde{B} = B_s(R - \hat{\lambda}Q_3)^{-1}B_s^*, \quad (4.14)$$

and

$$\tilde{C} = -\hat{\lambda}Q_1 - \hat{\lambda}^2 Q_2(R - \hat{\lambda}Q_3)^{-1}Q_2^*. \quad (4.15)$$

The variables satisfy a two point boundary value problem given by

$$\begin{pmatrix} \dot{x}_s \\ \dot{\rho} \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & -\tilde{A}^* \end{pmatrix} \begin{pmatrix} x_s \\ \rho \end{pmatrix}, \quad (4.16)$$

with

$$x_s(0) = 0, \quad \rho(T) = 0. \quad (4.17)$$

We now give a criterion for the estimation of $\hat{\lambda}$. Notice that $\hat{\lambda}$ gives a measure of performance of the H_∞ optimal controller under worst-case conditions.

THEOREM 4.2. Let $\hat{\lambda}$ be the smallest positive value for which the boundary value problem given by (4.16) and (4.17) has a solution (x_s, ρ) with $\int_0^T \{\frac{1}{2}x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2}v^* Q_3 v\} dt > 0$, where $v \triangleq (R - \hat{\lambda} Q_3)^{-1} \{\hat{\lambda} Q_2^* x_s + B_s^* \rho\}$. Then $\hat{\lambda}$ is the minimum value of (4.8), x_s is an optimal trajectory and $v = (R - \hat{\lambda} Q_3)^{-1} \{\hat{\lambda} Q_2^* x_s + B_s^* \rho\}$ is an open loop optimal exogenous input.

Proof. It is clear from Theorem 4.1 that if $v(t)$ minimizes (4.8), then it satisfies (4.16) and (4.17), with $\hat{\lambda}$ being the minimum value of (4.8). Now suppose $(x_s, \rho) \neq 0$ satisfies (4.16) and (4.17) for some $\hat{\lambda}$. Let $v = (R - \hat{\lambda} Q_3)^{-1} \{\hat{\lambda} Q_2^* x_s + B_s^* \rho\}$. In the following equations (\cdot, \cdot) denotes an inner product.

We have

$$\int_0^T ((R - \hat{\lambda} Q_3)v, v) dt = \int_0^T (\hat{\lambda} Q_2^* x_s, v) dt + \int_0^T (B_s^* \rho, v) dt. \quad (4.18)$$

By equation (4.7), the second integral of (4.18) can be written as

$$\int_0^T (B_s^* \rho, v) dt = \int_0^T (\rho, B_s v) dt = \int_0^T (\rho, \dot{x}_s - A_s x_s) dt. \quad (4.19)$$

An integration by parts and equations (4.10) and (4.17) yield

$$\int_0^T (\rho, \dot{x}_s - A_s x_s) dt = \hat{\lambda} \int_0^T (Q_1 x_s, x_s) dt + \hat{\lambda} \int_0^T (x_s, Q_2 v) dt. \quad (4.20)$$

Substituting (4.20) in (4.18), we get

$$\int_0^T v^* R v dt = \hat{\lambda} \int_0^T \{x_s^* Q_1 x_s + 2x_s^* Q_2 v + v^* Q_3 v\} dt. \quad (4.21)$$

Thus, the cost associated with v is $\hat{\lambda}$ provided that the right side of (4.21) is nonzero.

Hence, if (x_s, ρ) is a nontrivial solution of the boundary value problem given by (4.16) and

(4.17) for the smallest parameter $\hat{\lambda} > 0$ with $\int_0^T \{x_s^* Q_1 x_s + 2x_s^* Q_2 v + v^* Q_3 v\} dt > 0$, then $\hat{\lambda}$ is the optimal value and (x_s, ρ) is an optimal pair. \square

Note that the boundary value problem (4.16)-(4.17) has a solution with a nonvanishing denominator for (4.8) for at most a countably infinite values of $\hat{\lambda}$. Theorem 4.2 gives a sufficient condition for an open loop exogenous input to be optimal. Theorems 4.1 and 4.2 completely characterize the open loop worst case exogenous input.

Making use of the transition matrix, the solution of (4.16) may be expressed as

$$\begin{pmatrix} x_s(t) \\ \rho(t) \end{pmatrix} = \begin{pmatrix} \Phi_{11}(t, 0) & \Phi_{12}(t, 0) \\ \Phi_{21}(t, 0) & \Phi_{22}(t, 0) \end{pmatrix} \begin{pmatrix} x_s(0) \\ \rho(0) \end{pmatrix}. \quad (4.22)$$

Equation (4.17) yields

$$\Phi_{12}(T, 0)\rho(0) = x_s(T), \quad (4.23)$$

$$\Phi_{22}(T, 0)\rho(0) = 0. \quad (4.24)$$

In view of (4.24) and (4.16)-(4.17), we have $\det(\Phi_{22}(T, 0)) = 0$ if and only if the solution (x_s, ρ) of (4.16)-(4.17) is not identically zero. Thus, we need the least positive $\hat{\lambda}$ which makes $\det(\Phi_{22}(T, 0)) = 0$ and the denominator of (4.8) positive. This can be very easily obtained by doing a search with $\hat{\lambda}$ over an interval on which there is a change in the sign of the determinant.

We found the following algorithm to be numerically more stable since numbers of lesser magnitude are involved in the computation of the transition matrices in (4.25). We have

$$\begin{pmatrix} x_s(T) \\ \rho(T) \end{pmatrix} = \Phi(T, T/2)\Phi(T/2, 0) \begin{pmatrix} x_s(0) \\ \rho(0) \end{pmatrix}. \quad (4.25)$$

Let

$$\Phi^{-1}(T, T/2) = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix},$$

and

$$\Phi(T/2, 0) = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix}.$$

Making use of $x_s(0) = \rho(T) = 0$, we have

$$\nu_{12}\rho(0) = \xi_{11}x_s(T), \quad (4.26)$$

$$\nu_{22}\rho(0) = \xi_{21}x_s(T). \quad (4.27)$$

Thus

$$\det \begin{pmatrix} \nu_{12} & \xi_{11} \\ \nu_{22} & \xi_{21} \end{pmatrix} = 0. \quad (4.28)$$

Thus we need the least positive $\hat{\lambda}$ which makes the above determinant zero.

5. PERFORMANCE VARIATION

In this section, we develop a formula for the variation of $\hat{\lambda}$ when there are parameter variations in the system matrices. The system is given by

$$\dot{x}_s = A_s x_s + B_s v, \quad x_s(0) = 0, \quad (5.1)$$

with v chosen to minimize

$$\frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \left\{ \frac{1}{2} x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2} v^* Q_3 v \right\} dt}. \quad (5.2)$$

From the theory of Section 4, we have a boundary value problem given by

$$\dot{x}_s = \tilde{A} x_s + \tilde{B} \rho, \quad (5.3)$$

$$\dot{\rho} = \tilde{C} x_s - \tilde{A}^* \rho, \quad (5.4)$$

with

$$x_s(0) = 0, \quad \rho(T) = 0. \quad (5.5)$$

Note that $\hat{\lambda}$ is the minimal value of (5.2). Because of variations in the matrices of the original system (2.1)-(2.3), there will be corresponding variations in the matrices $\tilde{A}(t)$, $\tilde{B}(t)$, and $\tilde{C}(t)$. Let the elemental dependent variations in \tilde{A} , \tilde{B} , and \tilde{C} be denoted by $\delta\tilde{A}$, $\delta\tilde{B}$, and $\delta\tilde{C}$ respectively. For simplicity of notation, we will derive the variation in performance in terms of $\delta\tilde{A}$, $\delta\tilde{B}$, and $\delta\tilde{C}$. We denote by $\delta\hat{\lambda}$ the variation in $\hat{\lambda}$ owing to the parameter variations.

Let

$$\Upsilon = (R - \hat{\lambda}Q_3)^{-1}. \quad (5.6)$$

We have

$$\delta\tilde{A} = I_1 + \delta\hat{\lambda}J_1, \quad (5.7)$$

$$\delta\tilde{B} = I_2 + \delta\hat{\lambda}J_2, \quad (5.8)$$

$$\delta\tilde{C} = I_3 + \delta\hat{\lambda}J_3, \quad (5.9)$$

where

$$I_1 = \delta A_s + \hat{\lambda}\delta B_s \Upsilon Q_2^* + \hat{\lambda}B_s \Upsilon \delta R - \hat{\lambda}\delta Q_3 \Upsilon Q_2^* + \hat{\lambda}B_s \Upsilon \delta Q_2^*, \quad (5.10)$$

$$J_1 = -\hat{\lambda}B_s \Upsilon Q_3 \Upsilon Q_2^* + B_s \Upsilon Q_2^*, \quad (5.11)$$

$$I_2 = \delta B_s \Upsilon B_s^* + B_s \Upsilon \delta R - \hat{\lambda}\delta Q_3 \Upsilon B_s^* + B_s \Upsilon \delta B_s^*, \quad (5.12)$$

$$J_2 = -B_s \Upsilon Q_3 \Upsilon B_s^*, \quad (5.13)$$

$$I_3 = -\hat{\lambda}\delta Q_1 - \hat{\lambda}^2\delta Q_2 \Upsilon Q_2^* - \hat{\lambda}^2Q_2 \Upsilon \delta R - \hat{\lambda}\delta Q_3 \Upsilon Q_2^* - \hat{\lambda}^2Q_2 \Upsilon \delta Q_2^*, \quad (5.14)$$

$$J_3 = -Q_1 - 2\hat{\lambda}Q_2 \Upsilon Q_2^* + \hat{\lambda}^2Q_2 \Upsilon Q_3 \Upsilon Q_2^*. \quad (5.15)$$

Let x_1 and ρ_1 represent the variation in x_s and ρ due to $\delta\tilde{A}$, $\delta\tilde{B}$, and $\delta\tilde{C}$. From (5.3)-(5.5), we have the following equations that are satisfied by x_1 and ρ_1 .

$$\dot{x}_1 = \tilde{A}x_1 + \tilde{B}\rho_1 + (I_1 + \delta\hat{\lambda}J_1)x + (I_2 + \delta\hat{\lambda}J_2)\rho, \quad (5.16)$$

$$\dot{\rho}_1 = \tilde{C}x_1 - \tilde{A}^*\rho_1 + (I_3 + \delta\hat{\lambda}J_3)x - (I_1 + \delta\hat{\lambda}J_1)^*\rho, \quad (5.17)$$

$$x_1(0) = 0, \quad \rho_1(T) = 0. \quad (5.18)$$

THEOREM 5.1. The variation $\delta\hat{\lambda}$ in performance is given by

$$\delta\hat{\lambda} = \frac{-\int_0^T \{x_s^* I_1^* \rho + \frac{1}{2} \rho^* I_2 \rho - \frac{1}{2} x_s^* I_3 x_s\} dt}{\int_0^T \{\frac{1}{2} x_s^* Q_1 x_s + x_s^* Q_2 v + \frac{1}{2} v^* Q_3 v\} dt}, \quad (5.19)$$

where $v = \Upsilon\{B_s^* \rho + \hat{\lambda} Q_2^* x\}$.

Proof. From (5.17) we obtain

$$\begin{aligned} \int_0^T x_s^* \dot{\rho}_1 dt &= \int_0^T x_s^* \tilde{C}x_1 dt - \int_0^T x_s^* \tilde{A}^* \rho_1 dt \\ &\quad + \int_0^T x_s^* (I_3 + \delta\hat{\lambda}J_3)x_s dt - \int_0^T x_s^* (I_1 + \delta\hat{\lambda}J_1)^* \rho dt. \end{aligned} \quad (5.20)$$

Integrating the left side of (5.20) by parts and making use of (5.3) and (5.5), we obtain

$$-\int_0^T \rho_1^* \tilde{B} \rho dt = \int_0^T x_s^* \tilde{C}x_1 dt + \int_0^T x_s^* (I_3 + \delta\hat{\lambda}J_3)x_s dt - \int_0^T x_s^* (I_1 + \delta\hat{\lambda}J_1)^* \rho dt. \quad (5.21)$$

By (5.4), the first integral on the right side of (5.21) is written as

$$\int_0^T x_s^* \tilde{C}x_1 dt = \int_0^T (\dot{\rho} + \tilde{A}^* \rho)^* x_1 dt. \quad (5.22)$$

An integration by parts and equations (5.16) and (5.18) yield

$$\int_0^T x_s^* \tilde{C}x_1 dt = -\int_0^T \rho^* \tilde{B} \rho_1 dt - \int_0^T \rho^* (I_1 + \delta\hat{\lambda}J_1)x_s dt - \int_0^T \rho^* (I_2 + \delta\hat{\lambda}J_2)x_s dt. \quad (5.23)$$

Substituting (5.23) in (5.21) and simplifying, we obtain

$$\delta\hat{\lambda} = \frac{-\int_0^T \{2x_s^* I_1^* \rho + \rho^* I_2 \rho - x_s^* I_3 x_s\} dt}{\int_0^T \{2x_s^* J_1^* \rho + \rho^* J_2 \rho - x_s^* J_3 x_s\} dt}. \quad (5.24)$$

It can be easily shown (see Lemma 1 of Subrahmanyam 1992f, for example) that the denominator of (5.24) is equal to $\int_0^T \{x_s^* Q_1 x_s + 2x_s^* Q_2 v + v^* Q_3 v\} dt$. \square

6. PERFORMANCE ROBUSTNESS PROBLEM SOLUTION

We now give an iterative procedure to solve the performance robustness problem posed in Section 2. The equations are given by

$$\dot{x} = A(t)x + B_1(t)u + B_2(t)v, \quad x(0) = 0, \quad (6.1)$$

$$z = C(t)x + D(t)u + E(t)v, \quad (6.2)$$

$$y = C_2(t)x + D_2(t)u + E_2(t)v, \quad (6.3)$$

and

$$\lambda_{opt} = \max_u \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt}. \quad (6.4)$$

The suboptimal performance robustness problem is to find a controller, if one exists, for which

$$\min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \frac{1}{2} z^* W z dt} > \bar{\lambda}, \quad \bar{\lambda} < \lambda_{opt}, \quad (6.5)$$

under variations in the system matrices involved in (6.1)-(6.3).

The iterative steps are as follows.

1. Set $\lambda = \lambda_i$, $\lambda_i < \lambda_{opt}$, and design a suboptimal H_∞ controller based on the design equations of Section 3.

2. For this controller, let

$$\hat{\lambda} = \min_{v \neq 0} \frac{\int_0^T \frac{1}{2} v^* R v dt}{\int_0^T \left\{ \frac{1}{2} x^* Q_1 x + x^* Q_2 v + \frac{1}{2} v^* Q_3 v \right\} dt}. \quad (6.6)$$

This can be computed using the results of Section 4. Note that $\hat{\lambda} > \lambda$. If $\hat{\lambda} \leq \bar{\lambda}$, increase the value of λ and go through Steps 1 and 2 until $\hat{\lambda} > \bar{\lambda}$.

3. For the range of allowable variations of the matrices involved in (6.1)-(6.3), find the worst value of $\delta\hat{\lambda}$ using (5.19). Note that (5.19) is linear in the parameter variations. Thus, let $\delta = \min \delta\hat{\lambda}$. The controller of Step 1 solves the performance robustness problem in case $\hat{\lambda} + \delta > \bar{\lambda}$. If this is the case, stop the iteration and use the controller of Step 1.

4. If $\hat{\lambda} + \delta \leq \bar{\lambda}$, choose a suitable λ_{i+1} , $\lambda_{i+1} < \lambda_{opt}$, and set $\lambda = \lambda_{i+1}$. Go back to Step 1. Note that λ_{i+1} may be bigger or smaller than λ_i depending on the specific problem being solved.

Alternately we can get $\hat{\lambda}$ and $\delta = \min \delta\hat{\lambda}$ for a range of values of λ and then pick a value of λ that solves the performance robustness problem. In case we cannot find a value of λ which satisfies the performance robustness requirement, then the requirement is too stringent and the performance robustness problem does not have a solution. The requirement can be relaxed either by lowering $\bar{\lambda}$ or by suitably shrinking the range of allowable parameter variations.

7. AN EXAMPLE

We now illustrate the theory with an example. The system equations are given by

(6.1)-(6.4) with

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = (0 \ 1), \quad E = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ C_2 &= (1 \ 0), \quad D_2 = 0, \quad E_2 = (0 \ 1), \quad R = 100I_2, \quad W = I_2, \end{aligned}$$

where I_2 is the 2-dimensional identity matrix. Let

$$a_1 = \delta A(1, 1),$$

$$a_2 = \delta B_1(1, 1),$$

$$a_3 = \delta B_2(2, 1).$$

Assume that $a_i \in [-0.1, 0.1], i = 1, 2, 3$. Given a $\bar{\lambda}$, the problem is to find a controller for which

$$\min_{v \neq 0} \frac{\int_0^5 v^* R v \, dt}{\int_0^5 z^* W z \, dt} > \bar{\lambda} \quad (7.1)$$

under the above variations in $A(1, 1), B_1(1, 1)$ and $B_2(2, 1)$.

We can find λ_{opt} given by (2.4) using the technique given in Subrahmanyam 1992b,c,d. For this example, $\lambda_{opt} = 261.4016$. A suboptimal full state feedback controller can be designed choosing λ arbitrarily close to λ_{opt} . However, to obtain a viable suboptimal output feedback design, λ needs to be chosen much lower than λ_{opt} . Let us form Table 1 which gives the performance for various values of λ and the variations.

TABLE 1: Results of the Performance Robustness Example						
λ	$\hat{\lambda}$	$\delta\hat{\lambda}_1$	$\delta\hat{\lambda}_2$	$\delta\hat{\lambda}_3$	$\delta\hat{\lambda}_4$	$\hat{\lambda} - \delta$
1	5.4601	-0.7034	-3.0474	-0.0428	2.3868	2.4127
2	5.8783	-0.6757	-3.2158	0.0169	2.5232	2.6625
3	6.3303	-0.6428	-3.3935	0.0837	2.6670	2.9368
4	6.8179	-0.6045	-3.5808	0.1577	2.8186	3.2371
5	7.3431	-0.5605	-3.7778	0.2392	2.9781	3.5653
6	7.9079	-0.5107	-3.9847	0.3285	3.1455	3.9232
7	8.5140	-0.4549	-4.2010	0.4254	3.3207	4.3130
8	9.1630	-0.3933	-4.4254	0.5296	3.5026	4.7376
9	9.8562	-0.3258	-4.6543	0.6398	3.6886	5.2019
10	10.5944	-0.2534	-4.8779	0.7528	3.8717	5.7165

We now explain the procedure to obtain a particular row of the table. For example, for $\lambda = 3$, we find the output feedback controller using the equations given in Section 3. Note that we ignore the time dependence of the solutions (3.14) and (3.24) and only consider the constant solutions. For $\lambda = 3$,

$$\begin{aligned}
 U_1 P + U_2 &= (-2.4249 \quad -1.0062), \\
 V_1 P + V_2 &= \begin{pmatrix} 0.0302 & 0.0151 \\ 0 & 0 \end{pmatrix}, \\
 L &= \begin{pmatrix} -2.7524 \\ -0.3047 \end{pmatrix}.
 \end{aligned}$$

The closed loop system (4.1) is given by

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -2.4249 & -1.0062 \\ 0 & -1 & 0 & 0 \\ 2.7524 & 0 & -4.1773 & -0.0062 \\ 0.3047 & 0 & -0.2745 & -0.9849 \end{pmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2.7524 \\ 0 & 0.3047 \end{pmatrix} v, \quad (7.2)$$

with $x(0) = q(0) = 0$. Going through the procedure given in Section 4, the actual performance $\hat{\lambda}$ of the output feedback controller

$$u = (U_1 P + U_2)q \quad (7.3)$$

can be obtained as the first positive value of $\hat{\lambda}$ which satisfies (4.28). This value is $\hat{\lambda} = 6.3303$. Using (4.26)-(4.27) and selecting, for example, the first component of $\rho(0)$ as 1, we can obtain $\rho(0)$. Thus, we can solve the initial value problem (5.3)-(5.4) with $x_s(0) = 0$ and the value of $\rho(0)$ as obtained above.

Assuming that the accuracy of linear expressions is acceptable, $\delta\hat{\lambda}(a_1, a_2, a_3)$ which is the variation in performance can be obtained by using (5.19). Since (5.19) is linear in the parameter variations, the worst degradation in the value of $\hat{\lambda}$ is at one of the 8 vertices of (a_1, a_2, a_3) . Because of the linearity of $\delta\hat{\lambda}(a_1, a_2, a_3)$ and the symmetric nature of the variations, we need only compute $\delta\hat{\lambda}$ at four vertices. In Table 1,

$$\delta\hat{\lambda}_1 = \delta\hat{\lambda}(0.1, 0.1, 0.1),$$

$$\delta\hat{\lambda}_2 = \delta\hat{\lambda}(0.1, -0.1, 0.1),$$

$$\delta\hat{\lambda}_3 = \delta\hat{\lambda}(0.1, 0.1, -0.1),$$

$$\delta\hat{\lambda}_4 = \delta\hat{\lambda}(-0.1, 0.1, 0.1).$$

Also $\delta = \min_i \pm \delta\hat{\lambda}_i$. Thus $\hat{\lambda} - \delta$ in Table 1 gives the greatest lower bound of the actual performance of the controller under parameter variations. This value is 2.9368.

Going back to (7.1), if $\bar{\lambda} = 3$, looking at Table 1, λ can be 4 or higher. If $\bar{\lambda} = 4$, the controller design can be performed with $\lambda = 7$, or with a higher value of λ . If $\bar{\lambda} = 5$, we can select λ at 9 or higher.

8. CONCLUSIONS

In this technical note, an approach is presented for the solution of the finite horizon H_∞ performance robustness problem under parameter variations. A linear expression for

the degradation of performance is given in terms of variations in the system matrices. An example which illustrates the methodology is also given.

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